1. Introduction

A typical forward problem is "well-posed" in the sense that small changes in data cause small changes in the solution.

An electrocardiogram ECG produces a curve reflecting the electrical activity of the heart from measurements of electric potentials on the chest, and the curve gives the specialist information on abnormal activity of the heart such as abnormal heart rhythm (arrhythmias).
2. Inverse Problems

2.1. One-Dimensional Convection

\[ u'(x) = f(x) \text{ in } (0, 1], \quad u(0) = 0 \]  
with \( u : [0, 1] \rightarrow \mathbb{R} \).

Find \( f : [0, 1] \rightarrow \mathbb{R} \) which minimizes the total error

\[ J(f) = [u(1) - \bar{u}(1)]^2 + \mu \int_0^1 f(x)^2 \, dx \]

where \( \bar{u}(1) \) is the observed boundary value and \( \mu \geq 0 \) is a regularization constant.

Solution: \( f(x) = \frac{1}{1 + \mu} \bar{u}(1)x \) for \( x \in [0, 1] \)

2.2. One-Dimensional Diffusion

\[ -u''(x) = f(x) \text{ in } (0, 1), \quad u'(0) = 0, \quad u'(1) + u(1) = 0 \]

with \( u : [0, 1] \rightarrow \mathbb{R} \).

Find \( f(x) : [0, 1] \rightarrow \mathbb{R} \) which minimizes the total error

\[ J(f) = (u(0) - \bar{u}(0))^2 + (u(1) - \bar{u}(1))^2 + \mu \int_0^1 f(x)^2 \, dx \]

where \( \bar{u}(0), \bar{u}(1) \) are the observed boundary values and \( \mu \geq 0 \) is a regularization constant.

2.3. Poisson Equation

Given \( u(x) = \hat{U}(x) \) for \( x \in \Gamma \), find \( f(x) \) for \( x \in \Omega \)

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\partial_n u + \kappa u = 0 & \text{on } \Gamma
\end{cases}
\]

In discrete form as a least squares problem: Find \( F \in V_h \) which minimizes the objective function

\[ J(F) = ||U - \hat{U}||_1^2 + \mu ||F||_\Omega^2 \]

over \( V_h \), where \( U \in V_h \) satisfies

\[ (\nabla U, \nabla v)_\Omega + (\kappa U, v)_\Gamma = (F, v)_\Omega \]

for all \( v \in V_h \)

and \( V_h \) the space of continuous piecewise linear functions on \( \Omega \) of mesh size \( h(x) \).
2.4. Laplace Equation

Given $\bar{q} = \frac{\partial u}{\partial n}$ on $\Gamma_1$, find $f$ on $\Gamma_2$ such that

$$
\begin{cases}
-\Delta u &= 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \Gamma_0 \cup \Gamma_1 \\
u &= f \quad \text{on } \Gamma_2
\end{cases}
$$

(8)

We define $Bf = \frac{\partial u}{\partial n}$ on $\Gamma_1$. As a least squares problem: Find $f \in \Gamma_2$ which minimizes the objective function

$$J(f) = ||Bf - \bar{q}||_{\Gamma_1}^2 + \mu||f||_{\Gamma_2}^2$$

(9)

3. Integral Equations

3.1. What’s that?

There is a close correspondence between linear integral equations, which specify linear, integral relations among functions in an infinite-dimensional function space, and plain linear equations, which specify analogous relations among vectors in a finite-dimensional vector space.

Fredholm equations involve definite integrals with fixed upper and lower limits. An inhomogeneous Fredholm equation of the first kind has the form

$$\int_a^b K(t, s)f(s)ds = g(t)$$

(10)

Here $f(t)$ is the unknown function to be solved for, while $g(t)$ is a known "right-hand side”. The function of two variables, $K(t, s)$ is called the kernel.

3.2. Volterra Equation of 1st kind

$$g(t) = \int_a^t K(t, s)f(s)ds$$

(11)

3.3. Volterra Equation of 2nd kind

$$f(t) = \int_a^t K(t, s)f(s)ds + g(t)$$

(12)

Most algorithms for Volterra equations march out from $t = a$, building up the solution as they go. In this sense they resemble not only forward substitution, but also initial-value problems for ordinary differential equations. In fact many algorithms for ODE’s have counterparts for Volterra equations.

The simplest way to proceed is to solve the equation on a mesh with uniform spacing:

$$t_i = a + ih, \ i = 0, 1, \ldots, N, \ h \equiv \frac{b-a}{N}$$

(13)
To do so, a quadrature rule must be chosen. For a uniform mesh, the simplest scheme is the trapezoidal rule:

\[
\int_a^{t_i} K(t_i, s) f(s)ds = h \left( \frac{1}{2} K_{i0} f_0 + \sum_{j=1}^{i-1} K_{ij} f_j + \frac{1}{2} K_{ii} f_i \right)
\] (14)

Thus the trapezoidal method for equation (3.3)

\[
f_0 = g_0
\] (15)

\[
(1 - \frac{1}{2} h K_{ii}) f_i = h \left( \frac{1}{2} K_{i0} f_0 + \sum_{j=1}^{i-1} K_{ij} f_j \right) + g_i \quad i = 1, \ldots, N
\]

Equation (15) is an explicit prescription that gives the solution in \(O(N^2)\) operations. Unlike Fredholm equations, it is not necessary to solve a system of linear equations. Volterra equations thus involve less work than the corresponding Fredholm equations which do involve the inversion of, sometimes large, linear systems.

The efficiency of solving Volterra equations is somewhat counterbalanced by the fact that systems of these equations occur more frequently in practice. If we interpret equation (3.3) as a vector equation for the vector of \(m\) functions \(f(t)\), then the kernel \(K(t, s)\) is an \(m \times n\) matrix. Equation (15) must now also be understood as a vector equation. For each \(i\), we have to solve the \(m \times n\) set of linear algebraic equations by Gaussian elimination.

### 3.4. Fredholm Equation of 1st kind

\[
g(t) = \int_a^b K(t, s) f(s)ds
\] (16)

### 3.5. Fredholm Equation of 2nd kind

\[
f(t) = \lambda \int_a^b K(t, s) f(s)ds + g(t)
\] (17)

- Gaussian quadrature:

\[
f(t_i) = \lambda \sum_{j=1}^{N} w_j K(t_i, s_j) + g(t_i)
\] (18)

with \(\tilde{K}_{ij} = K_{ij} w_j\) in matrix form:

\[
(I - \lambda \tilde{K}) f = g
\] (19)

### 4. Optimization

#### 4.1. Positive Definiteness

- A symmetric matrix \(A \in \mathbb{R}^{n \times n}\) is positive definite if there is a positive constant \(\alpha\) such that
\[ x^T A x \geq \alpha \|x\|^2, \quad \text{for all } x \in \mathbb{R}^n. \] (20)

- A symmetric matrix \( A \in \mathbb{R}^{n \times n} \) is positive semidefinite if

\[ x^T A x \geq 0, \quad \text{for all } x \in \mathbb{R}^n. \] (21)

### 4.2. Convexity

- \( S \in \mathbb{R}^n \) is a **convex set** if the straight line segment connecting any two points in \( S \) lies entirely inside. Formally, for any two points \( x, y \in S \), we have

\[ \alpha x + (1 - \alpha) y \in S \quad \text{for all } \alpha \in [0, 1]. \] (22)

- \( f \) is a **convex function** if its domain is a convex set and if for any two points \( x, y \) in this domain, the graph of \( f \) lies below the straight line connecting \((x, f(x))\) to \((y, f(y))\) in the space \( \mathbb{R}^{n+1} \). That is, we have

\[ f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y), \quad \text{for all } \alpha \in [0, 1]. \] (23)

### 4.3. Local and global minima

- A point \( x^* \) is a **global minimizer** if \( f(x^*) \leq f(x) \) for all \( x \in \mathbb{R}^n \).

- A point \( x^* \) is a **local minimizer** if there is a neighborhood \( \mathcal{N} \) of \( x^* \) such that \( f(x^*) \leq f(x) \) for \( x \in \mathcal{N} \).

- A point \( x^* \) is a **strict local minimizer** (also called a **strong local minimizer**) if there is a neighborhood \( \mathcal{N} \) of \( x^* \) such that \( f(x^*) < f(x) \) for all \( x \in \mathcal{N} \) with \( x \neq x^* \).

- A point \( x^* \) is an **isolated local minimizer** if there is a neighborhood \( \mathcal{N} \) of \( x^* \) such that \( x^* \) is the only local minimizer in \( \mathcal{N} \).
Theorem 4.1 (Taylor’s Theorem) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and that $p \in \mathbb{R}^n$. Then we have that

$$f(x + p) = f(x) + \nabla f(x + tp)^T p,$$

for some $t \in (0, 1)$. Moreover, if $f$ is twice differentiable, we have that

$$\nabla f(x + p) = \nabla f(x) + \int_0^1 \nabla^2 f(x + tp)p dt,$$

and that

$$f(x + p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T \nabla^2 f(x + tp)p,$$

for some $t \in (0, 1)$.

Theorem 4.2 (First-Order Necessary Conditions) If $x^*$ is a local minimizer and $f$ is continuously differentiable in an open neighborhood of $x^*$, then $\nabla f(x^*) = 0$.

- $x^*$ is a stationary point if $\nabla f(x^*) = 0$.
- Any local minimizer must be a stationary point.

Theorem 4.3 (Second-Order Necessary Conditions) If $x^*$ is a local minimizer of $f$ and $\nabla^2 f$ is continuous in an open neighborhood of $x^*$, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive semidefinite.

Theorem 4.4 (Second-Order Sufficient Conditions) Suppose that $\nabla^2 f$ is continuous in an open neighborhood of $x^*$ and that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite. Then $x^*$ is a strict local minimizer of $f$.

Theorem 4.5 (Uniqueness of minimum for convex functions) When $f$ is convex, any local minimizer $x^*$ is a global minimizer of $f$. If in addition $f$ is differentiable, then any stationary point $x^*$ is a global minimizer of $f$.

References