A Note on the Computation of the Comparative Stress on RBC's

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The deviatoric stress tensor $T$ is related to the overall tensor $\sigma$ by: $T = \sigma - \frac{1}{3}(\text{tr}(\sigma))I$, where trace is defined in the following manner: $\text{tr}(\sigma) = \sigma_{11} + \sigma_{22} + \sigma_{33}$. The second invariant of a general tensor $A$ is defined as [Borsenko A and Tarapov I, 1968: Vector and Tensor Analysis]:

$$II_A = \frac{1}{2}(\text{tr}(A)^2 - \text{tr}(A^2)) \tag{1}$$

where for a 3-dimensional case:

$$\text{tr}(A)^2 = A_{11}^2 + A_{22}^2 + A_{33}^2 + A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33} + A_{33}A_{11} + A_{11}A_{22} \tag{2}$$

and

$$\text{tr}(A^2) = A_{11}^2 + A_{22}^2 + A_{33}^2 + A_{12}A_{21} + A_{13}A_{31} + A_{21}A_{12} + A_{23}A_{32} + A_{31}A_{13} + A_{32}A_{23} \tag{3}$$

from where for symmetrical $A$:

$$II_A = A_{11}A_{22} + A_{22}A_{33} + A_{33}A_{11} - A_{12}^2 - A_{23}^2 - A_{31}^2 \tag{4}$$

$$II_T =$$

$$= (\sigma_{11} - \sigma_m)(\sigma_{22} - \sigma_m) + (\sigma_{22} - \sigma_m)(\sigma_{33} - \sigma_m) + (\sigma_{33} - \sigma_m)(\sigma_{11} - \sigma_m) - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - 2\sigma_{11}\sigma_m - 2\sigma_{22}\sigma_m - 2\sigma_{33}\sigma_m + 3\sigma_m^2 - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - 6\sigma_m^2 + 3\sigma_m^2 - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - 3\sigma_m^2 - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - 3 \left( \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \right)^2 + 3\sigma_m^2 - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \frac{1}{3} \left( \sigma_{11} + \sigma_{12} + \sigma_{22} + \sigma_{23} + \sigma_{33} + \sigma_{11} \right) - \frac{1}{3} \left( \sigma_{12} + \sigma_{23} + \sigma_{31} \right)$$

$$= \frac{1}{3} \left( \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} \right) - \frac{1}{3} \left( \sigma_{11} + \sigma_{22} + \sigma_{33} \right) - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$= \frac{1}{6} \left( 2\sigma_{11}\sigma_{22} + 2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11} - 2\sigma_{11}^2 - 2\sigma_{22}^2 - 2\sigma_{33}^2 \right) - \left( \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right)$$

$$= -\frac{1}{6} \left[ ((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2) - (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right]$$
\[ -II_T = \]
\[ = \frac{1}{6} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \]
\[ = \frac{1}{6} \left[ ((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2) + 6 (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right] \]
\[ = \frac{1}{3} Y^2 = k^2 \]

\( Y \) is defined as tensile yield stress and \( k \) is the yield shear stress for a state of pure shear [see Johnson W. 1973: Engineering Plasticity].

\[ k = (-II_T)^{\frac{1}{2}} \]  

(5)

\[ \mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = 2\mu \mathbf{E}(u) \]  

(6)

\[ \mathbf{E}(u) = \frac{1}{2} (\nabla u + \nabla u^T) = \frac{1}{2} \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} \right) + \frac{1}{2} \left( \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} \right) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right) \]  

(7)

\[ \mathbf{E}(u) = \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & 2 \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & 2 \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} + \frac{\partial u}{\partial z} \end{pmatrix} \]  

(8)

\[ \mathbf{T} = \mu \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} + \frac{\partial u}{\partial y} \end{pmatrix} \]  

(9)

Additionally for an incompressible flow we have \( tr(\mathbf{T}) = 0 \) due to mass conservation:

\[ \nabla \cdot u = 0 \]  

(10)

\[ tr(\mathbf{T}) = 2\mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \]  

(11)

for \( tr(\mathbf{T})=0 \) the second invariant of \( \mathbf{T} \) is

\[ II_T = \]
\[ = -\frac{1}{2} tr(\mathbf{T}^2) \]
\[ = -\frac{1}{2} (T_{11}^2 + T_{22}^2 + T_{33}^2 + T_{12}T_{21} + T_{13}T_{31} + T_{21}T_{12} + T_{23}T_{32} + T_{31}T_{13} + T_{32}T_{23}) \]
\[ = -\frac{1}{2} \mathbf{T} : \mathbf{T} \]

Comparative shear stress as used by our group:

\[ \tau_{\text{scalar}} = (-II_T)^{\frac{1}{2}} = \left(\frac{1}{2} \mathbf{T} : \mathbf{T} \right)^{\frac{1}{2}} \]  

(12)
Formula for scalar stress by Bludszuweit [Bludszuweit C., 1997: Evaluation and Optimization of Artificial Organs by Computational Fluid Dynamics ASME FEDSM97-3424]:

$$\sigma_{scalar} = \left( \frac{1}{6} \sum (\sigma_{ii} - \sigma_{jj})^2 + \sum \sigma_{ij}^2 \right)^{\frac{1}{2}}$$  \hspace{1cm} (13)

The Ensight function ”fluid shear max” uses the same definition of shear stress.

Bludszuweit in her formula does not specify the indices of the sums but refers to [Johnson W., 1973: Engineering Plasticity].